

Friday, November 21, Lecture 35 - Calculus of Variations

- Last time ... Laplace transform

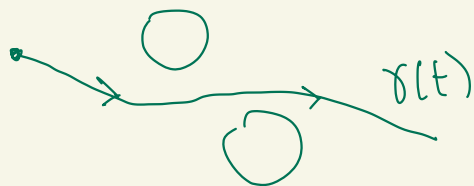
$$\mathcal{L}[f](s) = \int_0^{\infty} f(t) e^{-st} dt \quad s = \sigma + i\omega$$

exponential damping $\rightarrow e^{-\sigma t}$ $e^{-i\omega t}$ oscillations (Fourier Tx)

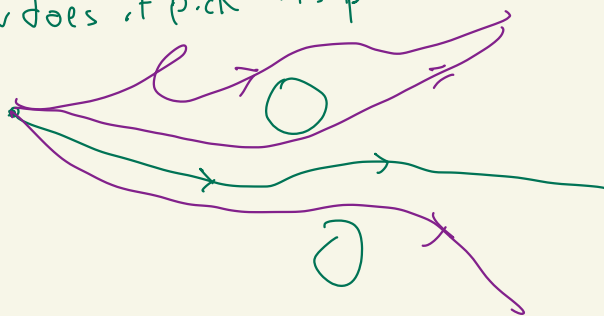
used to solve and characterize ODEs

Today:

Think about a particle moving in outer space.



How does it "pick" its path?



Principle of Least action: nature chooses the laziest path.

$\uparrow$
optimization

"Laziest" = minimize a suitably chosen function

$$I: \{\text{paths in space}\} \rightarrow \mathbb{R}$$

↑
this is a function
space

This is an optimization problem;

Once we have a good description of I ,

we can look for optimum by looking for
stationary points:

← this is the hard
part.

this isn't
too bad.

$$\frac{\partial I}{\partial v} = 0 \quad \text{for every direction } v.$$

(just like $\nabla I = 0$)

- Then, check to see if

minimum



maximum



neither



will use physical insights and perturbative analysis
to decide what the stationary point is.

Let's think about this function Γ and this space of functions. For simplicity, start with no other planets (no gravitational field).

We expect: particles move in straight lines (Thanks, Newton).

• $\rightarrow$

Landscape of paths:

$\gamma: [t_1, t_2] \rightarrow \mathbb{R}^3$ is a trajectory

$$t \mapsto \gamma(t)$$

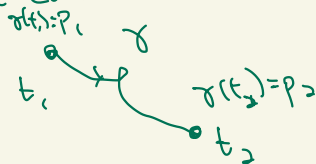


• Trajectories contain information both about γ and $\dot{\gamma}$, its first derivative.

To tell velocity information, need to know $\dot{\gamma}$

of course, this could also depend on higher derivatives, but these are the most important.

We can think of starting from a fixed path γ

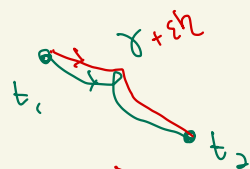


$\gamma(t_1) = \text{initial point } P_1$

$\gamma(t_2) = \text{final point } P_2$

and perturbing it by another path $\eta: [t_1, t_2] \rightarrow \mathbb{R}^3$

$\eta(t_1) = \eta(t_2) = 0$
(does not change endpoints)



if $\epsilon > 0$ is tiny,

γ is very close to $\gamma + \epsilon\eta$.

- Now, for "Laziest", let's choose the I which measures the length of the trajectory $\gamma: [0,1] \rightarrow \mathbb{R}^3$ $t_1=0, t_2=1$

$$I(\gamma) = \int_0^1 F(t, \gamma, \dot{\gamma}) dt$$

$$= \int_0^1 \|\dot{\gamma}\| dt$$

The arc length of γ .

- Now, for the perturbation.
Finite dim setting:



optimization for $f: \mathbb{R}^2 \rightarrow \mathbb{R}$,
minimum. Needs:

$$\left. \frac{\partial f}{\partial v} \right|_p = \left. \frac{\partial f(p + \epsilon v)}{\partial \epsilon} \right|_{\epsilon=0}$$

for every direction v .

space of functions:



optimization of
 $I: \{\text{trajectories}\} \rightarrow \mathbb{R}$

Needs:

$$\left. \frac{\partial I(\gamma + \epsilon h)}{\partial \epsilon} \right|_{\epsilon=0} = 0$$

For every direction h .

We call I stationary at γ if

$$\left. \frac{\partial I(\gamma + \epsilon h)}{\partial \epsilon} \right|_{\epsilon=0} = 0.$$

For every trajectory $h: [0,1] \rightarrow \mathbb{R}^3$
with $h(0)=0$ and $h(1)=1$.

Let's compute:

$$\begin{aligned} 0 &= \frac{dI(\gamma + \epsilon \eta)}{d\epsilon} \Big|_{\epsilon=0} = \int_0^1 \frac{d}{d\epsilon} \|\dot{\gamma} + \epsilon \dot{\eta}\| dt \\ &\quad \uparrow \\ &\text{Force th.s} \\ &\text{(so stationary)} \end{aligned} = \int_0^1 \frac{1}{2} \frac{1}{\|\dot{\gamma} + \epsilon \dot{\eta}\|} 2(\dot{\gamma} + \epsilon \dot{\eta}) \cdot \dot{\eta} dt \Big|_{\epsilon=0}$$
$$= \int_0^1 \frac{\dot{\gamma} \cdot \dot{\eta}}{\|\dot{\gamma}\|} dt$$

Integrate by parts:

$$\begin{aligned} 0 &= \frac{dI(\gamma + \epsilon \eta)}{d\epsilon} \Big|_{\epsilon=0} = \int_0^1 \frac{\dot{\gamma} \cdot \dot{\eta}}{\|\dot{\gamma}\|} dt \\ &= \cancel{\frac{\dot{\gamma} \cdot \eta}{\|\dot{\gamma}\|} \Big|_{t=0}^1} - \int_0^1 \eta \cdot \left(\frac{d}{dt} \frac{\dot{\gamma}}{\|\dot{\gamma}\|} \right) dt \end{aligned}$$

$\eta(0) = \eta(1) = 0$

This must hold for every perturbation η .

Lemma:

If $g: [0,1] \rightarrow \mathbb{R}^3$ has

$$0 = \int_0^1 \eta \cdot g dt$$

for every $\eta: [0,1] \rightarrow \mathbb{R}^3$, $\eta(0) = \eta(1) = 0$,

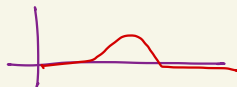
then g must be zero.

pf ideas:

$\therefore g \neq 0$, so



pick γ



$$\Rightarrow \int_0^1 g \cdot \dot{\gamma} dt \neq 0.$$

contradicts our assumption!



$$\Rightarrow \frac{d}{dt} \left(\frac{\dot{\gamma}}{\|\dot{\gamma}\|} \right) = 0$$

$$\Rightarrow \frac{\dot{\gamma}}{\|\dot{\gamma}\|} = C \quad \leftarrow \text{constant}$$

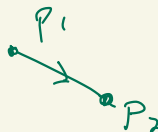
$$\begin{aligned} \gamma(0) &= p_1 \\ \gamma(1) &= p_2 \end{aligned}$$

in other words, the velocity $\dot{\gamma}$ has fixed direction:

$$\dot{\gamma} = C \|\dot{\gamma}\|$$

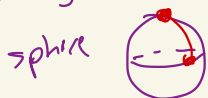
$\uparrow$ fixed vector $\uparrow$ scalar

must be pointing



So: this particle moves in a straight line!

we just verified that the geodesics - shortest paths - in $\mathbb{R}^3$ are straight lines.



sphere

cone:



space-time continuum

